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Super background field method for $N = 2$ SYM

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Abstract: The implementation of the Background Field Method (BFM) for quantum field theories is analysed within the Batalin-Vilkovisky (BV) formalism. We provide a systematic way of constructing general splittings of the fields into classical and quantum parts, such that the background transformations of the quantum fields are linear in the quantum variables. This leads to linear Ward-Takahashi identities for the background invariance and to great simplifications in multiloop computations. In addition, the gauge fixing is obtained by means of (anti)canonical transformations generated by the gauge-fixing fermion. Within this framework we derive the BFM for the $N = 2$ super-Yang-Mills theory in the Wess-Zumino gauge viewed as the twisted version of Donaldson-Witten topological gauge theory. We obtain the background transformations for the full BRST differential of $N = 2$ super-Yang-Mills (including gauge transformations, SUSY transformations and translations). The BFM permits all observables of the supersymmetric theory to be identified easily by computing the equivariant cohomology of the topological theory. These results should be regarded as a step towards the construction of a super BFM for the Minimal Supersymmetric Standard Model.

Keywords: Renormalization Regularization and Renormalons, Topological Field Theories, BRST Symmetry, Extended Supersymmetry.

Contents

1. Introduction

One of the most efficient techniques to perform computations in the framework of quantum field theory and string theory is the background field method (BFM). By introducing suitable classical background fields in the theory, it is possible to derive local Ward-Takahashi identities, which implement the background gauge transformations. The latter should be *linear* in quantum fields, in contrast to the BRST symmetry, which yields non-linear transformations of the quantum fields and, correspondingly, the Slavnov-Taylor (ST) identities for the quantum effective action.

As a consequence, the Ward-Takahashi (WT) identities for the background gauge invariance relate Green's functions at the same order of perturbation theory and they do not require the renormalization of those composite operators, associated to the BRST transformations, which are non-linear in the quantized fields.

At the level of the effective action, the background WT identities hold together with the ST ones, provided a suitable choice of a background gauge-invariant gauge-fixing has been performed. It turns out that the Green functions of physical BRST invariant operators can be computed by starting from the renormalized background gauge-invariant effective action, fulfilling the ST identities, after dropping the dependence on the quantum fields. The (physical) connected functions are then obtained by taking the Legendre transform with respect to the background fields, once a suitable gauge-fixing for the classical background

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fields is introduced [1, 2, 3]. It is this property, together with the advantages provided by the linearity of the WT identity in the process of renormalization [4, 5], that renders the BFM so appealing [6].

The fact that the correlation functions of gauge invariant observables can be equivalently computed within the BFM technique and with the conventional perturbative expansion (together with the conventional gauge fixing) can be expressed from the cohomological point of view by requiring that the dependence of Green functions upon the background fields be BRST trivial [1, 2, 3, 7]. This can be achieved by enlarging the BRST transformations to the background fields, in such a way that they form a set of BRST doublets together with their corresponding classical background ghosts. As a consequence, the BRST cohomology is unaffected by the presence of those new classical fields.

Such a procedure has been applied in [8, 5, 9, 10, 11, 12] to the case of models with closed algebras. In [8] the (non-linear) splitting of the scalar fields is achieved by using normal coordinates on the Riemann manifold [13], which leads to linear background gauge transformations of the quantum fields. On the other hand in [14] open gauge algebras within the Batalin-Vilkovisky (BV) formalism were studied, but the linear splitting of the fields is assumed from the beginning, so that it does not apply directly for example to those models where non-linear splittings are necessary in order to derive *linear* WT identities. This is the case for instance of $N = 2$ SYM, quantized in the Wess-Zumino gauge, or Donaldson-Witten theory, when one wishes to construct the background field transformations associated with the full BRST differential (including gauge transformations, supersymmetry transformations and translations).

In the Wess-Zumino gauge the supersymmetry transformations are non-linear. The most convenient way to handle the complete set of symmetries [15, 16] (gauge invariance, supersymmetry, R-symmetry, translations and Lorentz transformations) is to construct a generalized BRST operator fermionizing the parameters of the rigid symmetries. This leads to a set of ST identities, which is difficult to handle and to renormalize. For this reason, one would like to construct explicitly the background symmetry for the rigid and gauge symmetries of the model.

This requires a suitable change of variables, by which the original fields of the model are split into a background and a quantum part, with the requirement that the new quantum fields transform *linearly* under the background symmetry. If the existence of such a background symmetry can be established by solving the *splitting problem*, as we will discuss later, the classical action obeys the associated background WT identities.

In the case of supersymmetric theories the conventional BFM can be applied to implement background gauge invariance [17]. Applications of the gauge BFM in the superspace formalism have been discussed in [18]. However, the question of whether it is possible to extend the BFM to the full set of rigid and gauge symmetries — including SUSY transformations, R-symmetry, translations and Lorentz transformations — has to be studied.

At first, we clarify that *non-linear* splitting means that the relations (splitting functions) between the quantum fields and their backgrounds are characterized by complicated expressions involving higher order operators. Thus, the non-linear splittings are subject to modifications induced by radiative corrections and, consequently, they require new counterterms in perturbation theory. However, these functional relations can be constrained by symmetry requirements such as background gauge invariance and BRST symmetry and, finally, the linearity of the background transformations for quantum fields. If these conditions can be solved (existence of a solution of the splitting problem) classically, the corresponding WTI or STI would bring the same feature at the quantum level, namely the number of independent counterterms would be unchanged.

We first study the background splitting problem on a general ground in the BV formalism and we show how the antifields could help in the construction of the splitting functions. A by-product of this method is the implementation of the BFM by means of canonical transformations that guarantee that the physics of the model is not changed. In addition, also the rigid symmetries can be studied in the BV context by promoting the constant parameters of the rigid transformations to be constant ghosts.

In this paper, we briefly analyse the BFM in the Wess-Zumino model and present the BFM for the supersymmetry transformations within that model. Note that the Wess-Zumino model is an important element in the construction of the MSSM, as it represents both matter and Higgs sectors of the model. Moreover, after the elimination of the auxiliary fields, two supersymmetry transformations close on the equations of motion of the fermions. A naive application of the BFM requires an independent background action for the fermions. However, the latter is excluded by the invariance under the BRST transformation of the background fields Nevertheless, we show how this situation can be handled within the BV formalism by introducing a further field-antifield pair, which is required in order to take care of the closure in the background algebra.

We the consider $N = 2$ SYM in the Wess-Zumino (WZ) gauge (in the euclidean 4-dimensional spacetime). In this model the supersymmetry transformation of the gaugino is non-linear in the quantum fields, a feature also shared by $N = 1$ SYM in the WZ gauge [19]. The latter theory plays a distinguished rôle since it enters into the construction of the Minimal Supersymmetric Standard Model (MSSM) in the WZ gauge [15, 16], where most computations within MSSM have been carried out.

Having in mind $N = 1$ SYM, we will solve in this paper the splitting problem for $N = 2$ SYM, as a first step towards the study of $N = 1$ SYM. In the WZ gauge, $N = 2$ SYM presents some interesting features: the twisted formulation is equivalent to the Donaldson-Witten model. However, in the twisted theory the BRST differential has empty cohomology on the total space of polynomials in the fields and antifields [20]. This means that it is possible to find a redefinition of fields such that the BRST differential can be cast in the form of $s\mathcal{U} = \mathcal{V}$ and $s\mathcal{V} = 0$ of contractible pairs [20, 21, 22]. This simple form allows us to construct the linear splitting in the new variables and, mapping back to the original variables, the wanted non-linear splitting. The background gauge fixing is also studied and both the field redefinition and the gauge fixing procedure are achieved by means of canonical transformations.

A legitimate question is how to define the observables in the topological theory, in such a way that they can also be mapped back to the observables of the supersymmetric theory $N = 2$. Following [20], the observables are defined by computing the BRST cohomology in the space of polynomials with positive powers of the constant ghost ω (a twisted constant supersymmetric ghost). However, as pointed out in [23] the complete cohomology cannot be found in this way, and one has to impose further constraints. The main point is that the correct set of observables is identified in the topological version of the theory by means of the equivariant cohomology, as pointed out in [24]. Thus, one has to select the space of basic forms on whose space the BRST cohomology is computed. A practical method is to define the basic forms out of the complete space of local polynomials as the kernel of a new nilpotent anticommuting operator w which anti-commutes with the BRST symmetry. The new operator has been constructed in [24] and it turns out, by inspection, that it generates the background gauge transformations. Therefore, we conclude that observables are selected by computing the BRST cohomology on the space of background-invariant operators, which are independent of the background gauge ghost.

The paper is organized as follows. In section 2, we discuss the geometry of the splitting and extend the BRST symmetry to the background fields; in addition, we provide a general method, based on a cohomological analysis, to construct the BFM for a given model. This formulation, which relies on the BV formalism, can be applied to implement the BFM for generic models with field-dependent and open gauge algebras. In section 3, we apply the construction to the Donaldson-Witten model and to $N = 2$ SYM in the Wess-Zumino gauge. We construct the BFM for the full BRST differential, thus handling in the background formalism the full set of symmetries of the model (including gauge symmetries, SUSY transformations and translations). In section 3.2, the observables for $N = 2$ super-Yang-Mills are defined and the appendix contains some auxiliary material.

2. Geometry of BFM

2.1 Symmetries and non-linear splittings

We denote by ξ_i the quantum fields and by $\hat{\phi}_i$ their background partners. The original fields of the theory Φ_i are related to ξ_i and $\hat{\phi}_i$ by functions $\Phi_i = \Phi_i(\hat{\phi}_i, \xi_i)$. In the following we sometimes use the collective notation $\Phi = \{\Phi_i\}$, $\hat{\phi} = \{\phi_i\}$, and $\xi = \{\xi_i\}$.

At the classical level, the BRST transformations are described by the rules

$$s\,\Phi_i = R^\alpha_i[\Phi(\hat{\phi},\xi)]C_\alpha\,, \qquad s\,C_\alpha = \frac{1}{2}F^{\beta\gamma}_\alpha[\Phi(\hat{\phi},\xi)]C_\beta C_\gamma\,, \tag{2.1}$$

where C_α denote the ghost fields; $R^\alpha_i[\Phi(\hat{\phi},\xi)]$ are often assumed to be linear functions of the fields Φ. This requirement is fulfilled by many gauge theories for which the BFM has been implemented, as for instance Yang-Mills theory and the Standard Model. For the moment we limit ourselves to the case in which $F^{\beta\gamma}_\alpha$ are constant. $F^{\beta\gamma}_\alpha$ are antisymmetric in the $\beta\gamma$ indices, they are related to $R^\alpha_j[\Phi]$ by the algebra

$$R^\alpha_j[\Phi]\frac{\delta R^\beta_i[\Phi]}{\delta\Phi_j} - R^\beta_j[\Phi]\frac{\delta R^\alpha_i[\Phi]}{\delta\Phi_j} = F^{\alpha\beta}_\gamma R^\gamma_i[\Phi]\,, \tag{2.2}$$

and satisfy the Jacobi identities $F^{\beta[\gamma}_\alpha F^{\delta\sigma]}_\beta = 0$. In the next subsection we will also consider more general situations where $R^\alpha_i[\Phi(\hat{\phi},\xi)]$ and $F^{\beta\gamma}_\alpha[\Phi(\hat{\phi},\xi)]$ are polynomial expressions of the fields, and the algebra (2.1) is closed only on-shell.

For the background fields, we assign the following transformation rules

$$\begin{aligned} s\,\hat{\phi}_i &= \Omega_i + R_i^\alpha[\hat{\phi}]\hat{c}_\alpha\,, & s\,\hat{c}_\alpha &= \theta_\alpha + \frac{1}{2}F_\alpha^{\beta\gamma}\hat{c}_\beta\,\hat{c}_\gamma\,, \\ s\,\Omega_i &= \Omega_j\frac{\delta R_i^\alpha[\hat{\phi}]}{\delta\hat{\phi}_j}\hat{c}_\alpha - R_i^\alpha[\hat{\phi}]\theta_\alpha\,, & s\,\theta_\alpha &= F_\alpha^{\beta\gamma}\hat{c}_\beta\theta_\gamma\,, \end{aligned} \tag{2.3}$$

where $\hat{c}_\alpha$ are the backgrounds for the ghost fields associated to the background gauge symmetry. The new fields Ω_i and θ_α are introduced in order to control the dependence of the theory upon the background fields $\hat{\phi}_i$ and the background ghosts $\hat{c}_\alpha$. The BRST transformations in eq. (2.3) are nilpotent if the functions $R_j^\alpha[\hat{\phi}]$ are linear. It has been proven (see for example [25, 26, 27]) that the BRST cohomologies $H(s)$ and $H(s|d)$ are independent of the fields $\hat{\phi}_i, \hat{c}_\alpha, \Omega_i$ and θ_α and, therefore, the physical observables are not affected by the inclusion of such additional variables.

Notice that the structure of the BRST transformations for the background fields and the ghosts resemble the BRST symmetry for topological models [28]. This observation has been used in [29] to analyse the BRST cohomology for topological sigma models and will play a rôle in the forthcoming analysis.

The next step is to split the fields Φ_i into a quantum and a classical part

$$\Phi_i = \hat{\phi}_i + \Pi_i(\hat{\phi},\xi)\,, \tag{2.4}$$

where $\Pi_i(\hat{\phi},\xi) = O(\xi)$, in such a way that the background gauge transformations of the quantum fields ξ_i are linear in the quantum fields [8]. This leads to simple linear Ward-Takahashi identities for the Green functions. We start with the most general ansatz

$$s\,\xi_i = P_i^\alpha(\hat{\phi},\xi)C_\alpha + S_i^\alpha(\hat{\phi},\xi)\hat{c}_\alpha + Q_i^j(\hat{\phi},\xi)\Omega_j\,, \tag{2.5}$$

where $P_i^\alpha(\hat{\phi},\xi), Q_i^j(\hat{\phi},\xi)$ and $S_i^\alpha(\hat{\phi},\xi)$ are differential operators depending on background and quantum fields. The linearity condition for the background transformation of ξ_i yields $S_i^\alpha(\hat{\phi},\xi)$ to be linear in ξ_i. A linear splitting means that also $\Pi_i(\hat{\phi},\xi)$ in eq. (2.4) is at most linear in ξ_i, but as we anticipate, this is not always possible.

By inserting the ansatz (2.5) in the transformation rules (2.1), we derive the following equations

$$\begin{aligned} R_k^\alpha[\Phi(\hat{\phi},\xi)] &= P_i^\alpha(\hat{\phi},\xi)\frac{\delta\Pi_k}{\delta\xi_i}\,, \\ 0 &= \left(\delta_i^k + \frac{\delta\Pi_i}{\delta\hat{\phi}_k}\right) + \frac{\delta\Pi_i}{\delta\xi_l}Q_l^k(\hat{\phi},\xi)\,, \\ 0 &= \left(\delta_i^k + \frac{\delta\Pi_i}{\delta\hat{\phi}_k}\right)R_k^\alpha(\hat{\phi}) + \frac{\delta\Pi_i}{\delta\xi_k}S_k^\alpha(\hat{\phi},\xi)\,, \end{aligned} \tag{2.6}$$

which can be viewed either as consistency conditions for the functions $P_i^\alpha(\hat{\phi},\xi), Q_i^j(\hat{\phi},\xi)$ and $S_i^\alpha(\hat{\phi},\xi)$, given the splitting functions $\Pi_i(\hat{\phi},\xi)$, or as a constructon tool to obtain the splitting, by assuming the transformation rules (2.5).

For example, by eliminating the combination $(1+\delta\Pi/\delta\hat{\phi})$ from the second and the third equation, and by assuming that $\delta\Phi_i/\delta\xi_l$ is an invertible matrix, one has

$$S^\alpha_l(\hat{\phi},\xi) = Q^k_l(\hat{\phi},\xi) R^\alpha_k(\hat{\phi})\,, \tag{2.7}$$

which implies that also the function $Q^k_l(\hat{\phi},\xi)$ is linear in the quantum fields ξ_i.

In some cases $S^\alpha_i(\hat{\phi},\xi)$ turns out to be non-linear in the fields ξ_i. Moreover, it can happen that the BRST transformation of the ghost fields is non-linear in the fields ξ^i. In this case it is necessary to decompose also the original ghost fields C^α in (2.1) into $C^\alpha = \hat{c}^\alpha + \mathcal{G}^\alpha_\beta(\hat{\phi},\xi)\xi^\beta_C$ where $\hat{c}^\alpha$ is the background ghost introduced in (2.3). In order to respect the ghost number, the function $\mathcal{G}^\alpha_\beta(\hat{\phi},\xi)$ depends on zero-ghost number fields. Equation (2.5) now reads

$$s\,\xi_i = P^\alpha_i(\hat{\phi},\xi)\mathcal{G}_{\alpha\beta}(\hat{\phi},\xi)\xi^\beta_C + \mathcal{S}^\alpha_i(\hat{\phi},\xi)\hat{c}_\alpha + Q^j_i(\hat{\phi},\xi)\Omega_j\,, \tag{2.8}$$

where $\mathcal{S}^\alpha_i(\hat{\phi},\xi) = P^\alpha_i(\hat{\phi},\xi) + S^\alpha_i(\hat{\phi},\xi)$. The splitting of the ghost fields is chosen in such a way that $\mathcal{S}^\alpha_i(\hat{\phi},\xi)$ is a linear function of the quantum fields, namely of ξ.

For Yang-Mills theory [4], we can identify the symbols in eqs. (2.6) and (2.8) with the conventional notation: $\hat{\phi}_i \equiv \hat{A}^\alpha_\mu$, $\xi_i \equiv Q^\alpha_\mu$, $R^\alpha_k(\Phi) \equiv \nabla^a_\mu$ and $Q^k_l(\hat{\phi},\xi) \equiv \delta^a_b\delta^\mu_\nu$. It is easy to see that $\Pi^a_\mu(\hat{A},Q) = Q^a_\mu$ solves eq. (2.6) and eq. (2.7). Notice that there is a more general solution to eqs. (2.6) with $\Pi^a_\mu(\hat{A},Q) = \Theta^{ab}_{\mu\nu}(\hat{A})Q^\nu_b$, where $\Theta^{ab}_{\mu\nu}(\hat{A})$ is a combination of background gauge invariant operators. From equation (2.3) we see that

$$s\hat{A}^a_\mu = \Omega^a_\mu + \partial_\mu \hat{c}^a + f^{abc}\hat{A}^b_\mu \hat{c}^c\,. \tag{2.9}$$

Then from eq. (2.1) we obtain

$$sQ^a_\mu = \partial_\mu(C^a - \hat{c}^a) + f^{abc}\hat{A}^b_\mu(C^c - \hat{c}^c) + f^{abc}Q^b_\mu C^c - \Omega^a_\mu\,. \tag{2.10}$$

By splitting the original ghost C^a into $C^a = \hat{c}^a + \xi^a_C$ we can rewrite (2.10) in the following way

$$sQ^a_\mu = \partial_\mu \xi^a_C + f^{abc}\hat{A}^b_\mu \xi^c_C + f^{abc}Q^b_\mu \xi^c_C + f^{abc}Q^b_\mu \hat{c}^c - \Omega^a_\mu\,. \tag{2.11}$$

The background transformation of Q^a_μ has to be identified with the fourth term in the above equation, which is of order 1 in the ghost background $\hat{c}^a$. This leads to the identification $\mathcal{S}^\alpha_i(\hat{\phi},\xi) \equiv f^{abc}Q^b_\mu$. Moreover, the third term is bilinear in the quantum fields and $P^\alpha_i(\hat{\phi},\xi)\mathcal{G}_{\alpha\beta}(\hat{\phi},\xi) \equiv \hat{\nabla}^{ab}_\mu - f^{abc}Q^c_\mu$. A suitable choice of the splitting of ghost fields simplifies the structure of quantum gauge transformations.

For non-linear sigma models (cf. [8]), the gauge transformations (2.1) for the coordinates Φ_i of the manifold are replaced by diffeomorphisms $R^\alpha_i(\Phi)C_\alpha \to v_i(\Phi)$, where v_i are the components of a vector field[1] and eqs. (2.6) are rewritten in the form

$$v_i(\Phi) = \left(\delta^k_i + \frac{\delta\Pi_i}{\delta\hat{\phi}_k}\right) v_k(\hat{\phi}) + \xi_l\partial^l v_k(\hat{\phi})\frac{\delta\Pi_i}{\delta\xi_k}$$

[1]The transformations $\delta\Phi_i = v_i(\Phi)$ are rigid transformations from the worldsheet point of view. They can be translated into BRST transformations by decomposing $v(\Phi)$ into power series and fermionizing the coefficients: $v(\Phi) = \sum_n v_n\Phi^n \to \sum_n c_n\Phi^n$ where c_n are an infinite set of constant anticommuting ghosts. Then, we have $s\,\Phi_i = \sum_n c_n\Phi^n$ and $s\,c_n = \sum_m (m-n)c_{m-n}c_n$. The latter are the usual BRST transformations for the ghost fields for the Virasoro algebra.

$$0=\left(\delta_i^k+\frac{\delta\Pi_i}{\delta\hat{\phi}_k}\right)+\frac{\delta\Pi_i}{\delta\xi_l}Q_l^k(\hat{\phi},\xi)\,, \tag{2.12}$$

and $S_i(\hat{\phi},\xi)=\xi_k\partial^k v_i(\hat{\phi})$.[2] Following [8], we can use an interpolating field $\Phi_i(t)$, which satisfies the geodesic equations for a given connection Γ^i_{jk} and construct a solution to (2.12)

$$\begin{aligned}\Phi_i(\hat{\phi},\xi)&=\hat{\phi}_i+\Pi_i(\hat{\phi},\xi)\,,\qquad \Pi_i(\hat{\phi},\xi)=\xi_i+\chi_i(\hat{\phi},\xi)\,,\\ \chi_i(\hat{\phi},\xi)&=-\sum_{n=2}^{\infty}\frac{1}{n!}\Gamma_i^{j_1\dots j_n}(\hat{\phi})\xi_{j_1}\dots\xi_{j_n}\,,\end{aligned} \tag{2.13}$$

where $\Gamma_i^{j_1\dots j_n}(\hat{\phi})$ are related to the covariant derivatives of the connection Γ^i_{jk} computed at the point $\hat{\phi}_i$. In this example we can easily justify the invertibility of the matrix $\frac{\delta\Phi_i}{\delta\xi_j}$; in fact, we have

$$\frac{\delta\Phi_i}{\delta\xi_j}=\delta_i^j+O(\xi) \tag{2.14}$$

and, by equations. (2.13), $\delta\Phi_i/\delta\xi_j$ is invertible as a formal power series. Equation (2.13) provides an explicit and particular solution for the splitting. This is not the only possible solution compatible with a linear transformation of the quantum field $\mathcal{S}_i(\hat{\phi},\xi)=\xi_k\partial^k v_i(\hat{\phi})$. However we point out that (2.13) turns out to be the most general solution (up to rotations in the ξ-space) for the background transformation of the quantum field.

Following these suggestions of the non-linear sigma model, we can deduce that the most general solution for $\mathcal{S}_i^\alpha(\hat{\phi},\xi)$ is given by

$$\mathcal{S}_i^\alpha(\hat{\phi},\xi)=\frac{\delta R_i^\alpha(\hat{\phi})}{\delta\hat{\phi}_j}\xi_j\,. \tag{2.15}$$

To prove this assertion, we insert equation (2.4) into the third of equations (2.6) and we expand up to the first order in ξ_j. We get

$$\begin{aligned}R_k^\alpha[\hat{\phi}]+\frac{\delta R_k^\alpha[\hat{\phi}]}{\delta\hat{\phi}_j}\frac{\delta\Pi_j}{\delta\xi_p}\bigg|_{\xi=0}\xi_p+\cdots&=R_k^\alpha[\hat{\phi}]+\frac{\delta^2\Pi_k}{\delta\hat{\phi}_i\delta\xi_p}\bigg|_{\xi=0}\xi_p R_i^\alpha[\hat{\phi}]\,\frac{\delta\Pi_k}{\delta\xi_j}\bigg|_{\xi=0}S_j^\alpha|_{\xi=0}+\\ &+\frac{\delta\Pi_k}{\delta\xi_j}\bigg|_{\xi=0}\frac{\delta S_j^\alpha}{\delta\xi_p}\bigg|_{\xi=0}\xi_p+\frac{\delta^2\Pi_k}{\delta\xi_i\delta\xi_p}\bigg|_{\xi=0}\xi_p S_i^\alpha|_{\xi=0}+\cdots\,.\end{aligned} \tag{2.16}$$

By looking at the terms of order zero in ξ we get

$$R_k^\alpha[\hat{\phi}]=R_k^\alpha[\hat{\phi}]+\frac{\delta\Pi_k}{\delta\xi_j}\bigg|_{\xi=0}S_j^\alpha|_{\xi=0}\,. \tag{2.17}$$

This gives $S_j^\alpha|_{\xi=0}=0$ since $\frac{\delta\Pi_k}{\delta\xi_j}|_{\xi=0}$ is invertible. Then we look at the terms of order one in ξ in equation (2.16), by taking into account $S_j^\alpha|_{\xi=0}=0$ and the fact that $\frac{\delta^2\Pi_k}{\delta\hat{\phi}_i\delta\xi_p}|_{\xi=0}=0$

[2] We remind the reader that if ξ is a vector field, it is natural to define $\delta\xi=\mathcal{L}_v\xi$, where $\mathcal{L}_v$ is the Lie derivative. This means that $\delta(\xi_i\partial^i)=(\xi_k\partial^k v_i-v_k\partial^k\xi_i)\partial^i$. Assuming that ξ_i are constant and independent of $\hat{\phi}_i$, we have $\delta\xi_i=\xi_k\partial^k v_i$.

because of the invertibility of $\frac{\delta \Pi_k}{\delta \xi_j}$ as a formal power series. We obtain

$$\frac{\delta R_k^\alpha[\hat{\phi}]}{\delta \hat{\phi}_j} \frac{\delta \Pi_j}{\delta \xi_p}\bigg|_{\xi=0} = \frac{\delta \Pi_k}{\delta \xi_j}\bigg|_{\xi=0} \frac{\delta S_j^\alpha}{\delta \xi_p}\bigg|_{\xi=0}, \tag{2.18}$$

and finally

$$\frac{\delta S_i^\alpha}{\delta \xi_p}\bigg|_{\xi=0} = \frac{\delta \xi_i}{\delta \Pi_k}\bigg|_{\xi=0} \frac{\delta R_k^\alpha[\hat{\phi}]}{\delta \hat{\phi}_j} \frac{\delta \Pi_j}{\delta \xi_p}\bigg|_{\xi=0}. \tag{2.19}$$

This means that up to a rotation in the ξ-space we recover equation (2.15).

2.2 BV formulation of the splitting problem

The main question is how to solve equations (2.6) or equations (2.12) to find the splitting for the fields Φ_i and for the ghost fields C^α.

For this purpose, we rewrite equations (2.6) in a different form, suitable for the direct application of the BV formalism and for a cohomological reformulation of the splitting problem. We work out the necessary formalism for the general case of an open algebra. It is convenient to introduce the antifields for each field of the model and to modify the classical gauge invariant action S_0 into $S = S_0 + S_*$, where S_* is given by

$$S_* = \int \left(\Phi^{*,i} s\, \Phi_i + C^{*,\alpha} s\, C_\alpha + \hat{\phi}^{*,i} s\, \hat{\phi}_i + \hat{c}^{*,\alpha} s\, \hat{c}_\alpha + \Omega^{*,i} s\, \Omega_i + \theta^{*,\alpha} s\, \theta_\alpha \right). \tag{2.20}$$

S obeys the master equation

$$(S, S) = 0\,. \tag{2.21}$$

The bracket in equation (2.21) is defined by

$$(X, Y) \equiv \int \left(\frac{\delta_r X}{\delta \varphi^I} \frac{\delta_l Y}{\delta \varphi^*_I} - \frac{\delta_r X}{\delta \varphi^*_I} \frac{\delta_l Y}{\delta \varphi^I} \right), \tag{2.22}$$

where $\varphi^I = \{\Phi^i, C^\alpha, \hat{\phi}^i, \hat{c}_\alpha, \Omega_i, \theta_\alpha\}$, $\varphi^*_I = \{\Phi^{*,i}, C^{*,\alpha}, \hat{\phi}^{*,i}, \hat{c}^{*,\alpha}, \Omega^{*,i}, \theta^{*,\alpha}\}$. In principle, one should not need antifields for classical fields such as the background fields $\hat{\phi}_i$ and for their shifts Ω_i, but it turns out that they are needed in order to handle open algebras. Indeed, with antifields, one can easily accommodate general gauge algebras of the form

$$R_j^\alpha \frac{\delta R_i^\beta}{\delta \Phi_j} - R_j^\beta \frac{\delta R_i^\alpha}{\delta \Phi_j} = F_\gamma^{\alpha\beta}[\Phi] R_i^\gamma + M_{ij}^{\alpha\beta}[\Phi] \frac{\delta S_0}{\delta \Phi_j}, \tag{2.23}$$

where $F_\gamma^{\alpha\beta}[\Phi]$ and $M_{ij}^{\alpha\beta}[\Phi]$ involve dynamical variables Φ_i. The algebra described by the generators $R_j^\alpha[\Phi]$ is an open algebra and the last term in (2.23) takes into account those symmetries which are closed on the classical equations of motion $\delta S_0/\delta \Phi_j = 0$. The latter term is not there in the case of a closed algebra of course. By consistency with the invariance of action S_0, one finds that $M_{ij}^{\alpha\beta}[\Phi] = -M_{ji}^{\alpha\beta}[\Phi]$ and $M_{ij}^{\alpha\beta}[\Phi] = -M_{ij}^{\beta\alpha}[\Phi]$.

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By using the antifields, the BRST transformations are modified into

$$s\,\Phi_i = R_i^\alpha[\Phi]\,C_\alpha + M_{ij}^{\beta\alpha}[\Phi]\,C_\alpha C_\beta \Phi^{*,j}\,, \qquad s\,\Phi^{*,i} = \frac{\delta S}{\delta \Phi_i}\,. \tag{2.24}$$

The fulfilment of the master equation in equation (2.21) requires that the action be changed by adding new terms that are quadratic in the antifields

$$S \to S + \frac{1}{2}\int M_{ij}^{\beta\alpha}[\Phi] C_\alpha C_\beta \Phi^{*,i}\Phi^{*,j}\,. \tag{2.25}$$

The nilpotency of the BRST transformation on the antifield $\Phi^{*,j}$ then follows from the invariance of the action S_0 and of the antifield terms S_*.

In more general cases, for example in case of reducible gauge theories, one usually needs new terms with higher powers of antifields and new ghost fields to parametrize the new symmetries.

We notice that, corresponding to the symmetry (2.24), we can introduce a background gauge symmetry, where we replace Φ and Φ^* with the background partners everywhere and the ghost C_α with the background ghost $\hat{c}_\alpha$. In addition, we still have to add the shift fields generated by Ω_i and θ_α. At first one might think that the natural definition of the background symmetry for the background fields is:

$$s\,\hat{\phi}_i = R_i^\alpha[\hat{\phi}]\,\hat{c}_\alpha + M_{ij}^{\beta\alpha}[\hat{\phi}]\,\hat{c}_\alpha \hat{c}_\beta \hat{\phi}^{*,j} + \Omega_i\,, \qquad s\,\hat{c}_\alpha = \frac{1}{2}F_\alpha^{\beta\gamma}[\hat{\phi}]\hat{c}_\beta \hat{c}_\gamma + \theta_\alpha\,. \tag{2.26}$$

However, this would lead to a difficulty: in order to reproduce, for the background fields, the same gauge algebra as in equation (2.24), we should add to the classical action S new terms in order to generate the "closure terms,, proportional to the equations of motion. This is excluded by the presence of the shifts generated by the fields Ω_i and θ_α. In order to circumvent this problem we introduce new antifields, denoted by $\hat{\chi}^{*,j}$, which replace the antifields $\hat{\phi}^{*,j}$ in equations (2.26)

$$s\,\hat{\phi}_i = R_i^\alpha[\hat{\phi}]\,\hat{c}_\alpha + M_{ij}^{\beta\alpha}[\hat{\phi}]\,\hat{c}_\alpha \hat{c}_\beta \hat{\chi}^{*,j} + \Omega_i\,, \qquad s\,\hat{c}_\alpha = F_\gamma^{\alpha\beta}[\hat{\phi}] R_i^\gamma[\hat{\phi}] + \theta_\alpha\,. \tag{2.27}$$

Their transformation rules reproduce the correct algebra by imposing

$$s\,\hat{\chi}^{*,i} = \left(\frac{\delta S_0[\Phi]}{\delta \Phi_i}\right)_{\Phi\to\hat{\phi}} + \Omega_\chi^{*,i} + \cdots\,, \tag{2.28}$$

where $(\delta S_0/\delta\Phi_i)_{\Phi\to\hat{\phi}}$ is the classical gauge covariant equations of motion where the original fields Φ_i have been replaced by the background fields $\hat{\phi}_i$, $\Omega_\chi^{*,i}$ are the shift fields for $\hat{\chi}^{*,i}$ and the ellipsis denotes further terms with at least one power of antifields, eventually needed to guarantee the closure of the algebra.

The requirement of nilpotency of s on $\hat{\phi}_i$ imposes a constraint on $s\Omega_i$, while nilpotency on $\hat{\chi}^{*,i}$ yields a constraint for $s\Omega_\chi^{*,i}$.

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Finally, we can introduce the BRST transformations for the quantum fields ξ_i and their antifields ξ_i^*. The only assumption we have to impose here is that the quantum fields should transform linearly under the background gauge transformations

$$
\begin{aligned}
s\,\xi_k &= \xi_l \left.\frac{\delta R_k^\alpha}{\delta \Phi_l}\right|_{\Phi\to\hat{\phi}} \hat{c}_\alpha + \frac{1}{2}\hat{c}_\alpha \hat{c}_\beta N_{kj}^{\alpha\beta}[\hat{\phi}]\,\xi^{*,j} + \cdots \\
s\,\xi^{*,k} &= \xi_l \left.\frac{\delta S_0}{\delta \Phi_l \delta \Phi_k}\right|_{\Phi\to\hat{\phi}} + \cdots
\end{aligned}
\tag{2.29}
$$

where the ellipses denote the BRST transformations generated by the quantum ghosts and by the shifts Ω_i. In order to compute these remaining terms of rules (2.28) and (2.29), one has to solve the master equation (2.21) with the bracket given in terms of

$$
\varphi^I = \{\xi^i, \xi_C^\alpha, \hat{\phi}^i, \hat{c}_\alpha, \Omega_i, \theta_\alpha, \hat{\chi}_i, \Omega_{\chi^i}\}, \qquad \varphi_I^* = \{\xi^{*,i}, \xi_C^{*,\alpha}, \hat{\phi}^{*,i}, \hat{c}^{*,\alpha}, \Omega^{*,i}, \theta^{*,\alpha}, \hat{\chi}^{*,i}, \Omega_\chi^{*,i}\}
$$

(ξ_C^α and $\xi_C^{*,\alpha}$ are the quantum ghost fields and their "quantum" antifields, respectively), with boundary conditions (2.27), (2.28) and (2.29). It is easy to check that in the case of the Yang-Mills theory and the non-linear sigma model, this leads to the usual splitting between quantum and classical fields. In more general cases, one has to show that there is at least a solution.

Notice that since all the fields and antifields in the background sector are classical fields, there is no distinction between a field and an antifield from the point of view of the quantization of the model. $\Omega^{*,i}$ removes the reducibility between the "quantum,, antifield $\xi^{*,i}$ and the classical antifield $\hat{\phi}^{*,i}$.

As an illustration of the previous considerations, we analyse a simple model, namely the $N=1$ Wess-Zumino model [30]. There, the role of the antifields $\hat{\chi}^{*,i}$ and their shifts $\Omega_\chi^{*,i}$ will become clear. In this case the antifields $\hat{\chi}^{*,i}$ can be interpreted as the background counterpart of the auxiliary field F. The same thechnique has been used in [15].

The model is written in terms of the fields $\Phi_i = \{A, \psi^\alpha\}$, where A is a complex scalar field and ψ^α is a Weyl spinor.[3] We also introduce the ghost fields η^α for the supersymmetry and v^μ for the translations. Since there is no gauge symmetry, we consider only rigid supersymmetry transformations. By eliminating the auxiliary fields, the algebra of supersymmetry closes only on-shell. The problem can be reformulated in the context of the BV framework. The classical action $S = S_0 + S_1 + S_2$ reads

$$
\begin{aligned}
S_0 &= \int d^4x \left(|\partial_\mu A|^2 - i\psi^\alpha \sigma^\mu_{\alpha\dot{\beta}} \partial_\mu \bar{\psi}^{\dot{\beta}} \right) , \\
S_1 &= \int d^4x \left(A^*(sA) + \psi^{*\alpha}(s\psi_\alpha) + v^{*\mu}(sv_\mu) + \text{c.c.} \right) ,
\end{aligned}
$$

[3] The motivation to consider a BFM formulation for the Wess-Zumino models is related to the implementation of the BFM for the MSSM. There two Wess-Zumino models for the Higgs superfields H_1 and H_2 are coupled to the gauge invariant action in order to break the $\mathrm{SU}(2)_L \times U_Y(1)$ down to the subgroup $U_Q(1)$. In order to write a generalization of the 't Hooft-background gauge fixing for the MSSM, one needs to add the background fields for the scalar components of H_1 and H_2. In addition, in order to mantain the supersymmetry manifest, one has to add also the background for their superpartners.

$$S_2 = \int d^4x \left(2\eta^\alpha \psi^*_\alpha \, \bar\eta_{\dot\beta} \bar\psi^{*\dot\beta} \right) , \tag{2.30}$$

where the BRST transformations are given by

$$\begin{aligned} s\,A &= 2\eta^\alpha \psi_\alpha - i v^\mu \partial_\mu A \,, \qquad s\,\psi_\alpha = -i\sigma^\mu_{\alpha\dot\beta} \bar\eta^{\dot\beta} \partial_\mu A - i v^\mu \partial_\mu \psi_\alpha - 2\eta^\alpha \, \bar\eta_{\dot\beta} \bar\psi^{*\dot\beta} \,, \\ s\,v^\mu &= -2\eta^\alpha \sigma^\mu_{\alpha\dot\beta} \bar\eta^{\dot\beta} \,, \qquad s\,\eta^\alpha = 0 \,. \end{aligned} \tag{2.31}$$

The BRST transformation of the fermion ψ_α contains the antifield $\bar\psi^{*\dot\beta}$ in order to take into account the closure on the equations of motion. This is reflected at the level of the classical action in the term S_2, quadratic in the antifields. We can derive the BRST transformations for the background field as

$$\begin{aligned} s\,\hat A &= 2\hat\eta^\alpha \hat\psi_\alpha - i\hat v^\mu \partial_\mu \hat A + \Omega_A \,, \\ s\,\hat\psi_\alpha &= -i\sigma^\mu_{\alpha\dot\beta} \hat{\bar\eta}^{\dot\beta} \partial_\mu \hat A - i\hat v^\mu \partial_\mu \hat\psi_\alpha - 2\hat\eta_\alpha \, \hat F + \Omega_{\psi\,\alpha} \,, \\ s\,\hat v^\mu &= -2\hat\eta^\alpha \sigma^\mu_{\alpha\dot\beta} \hat{\bar\eta}^{\dot\beta} + \theta^\mu_v \,, \\ s\,\hat\eta^\alpha &= \theta^\alpha \,. \end{aligned} \tag{2.32}$$

qThe fields Ω_A and Ω^α_ψ are the shift for the background fields $\hat A$ and $\hat\psi_\alpha$ and the fields θ^α and θ^μ_v are the background fields for the ghost fields $\hat\eta_\alpha$ and $\hat v^\mu$. In the above equation we have reintroduced the auxiliary field $\hat F$. The BRST transformation rules are given by

$$s\,\hat F = -i\hat{\bar\eta}^{\dot\beta} \bar\sigma^\mu_{\dot\beta\alpha} \partial_\mu \hat\psi^\alpha - i\hat v^\mu \partial_\mu \hat F + \Omega_F \,, \tag{2.33}$$

where Ω_F is the corresponding shift field. A simple exercise shows that, in order to mantain the nilpotency on $\hat\psi_\alpha$, one needs to impose the following transformation:

$$s\,\Omega_{\psi\,\alpha} = 2\hat\eta_\alpha \Omega_F + i\hat v^\mu \partial_\mu \Omega_{\psi\,\alpha} + i\sigma^\mu_{\alpha\dot\beta} \hat{\bar\eta}^{\dot\beta} \partial_\mu \bar\Omega_A + 2\theta_\alpha \hat F + i\theta^\mu_v \partial_\mu \hat\psi_\alpha + i\sigma^\mu_{\alpha\dot\beta} \bar\theta^{\dot\beta} \partial_\mu \hat A \,, \tag{2.34}$$

which represents the supersymmetry algebra at the level of the fields $\Omega^\alpha_\psi, \Omega_F, \Omega_A$. They form a chiral multiplet.[4]

We can avoid the introduction of the auxiliary fields by using an additional antifield, as outlined before. In order to distinguish the antifield $\hat\psi^{*,\alpha}$ (coupled to the BRST variation of ψ^α) from the new antifield, which is needed to reproduce the correct algebra at the level of the background fields, we will denote the latter by $\hat\chi^{*,\alpha}$.

The BRST transformation for the spinor ψ_α is correspondingly given by

$$s\,\hat\psi_\alpha = -i\sigma^\mu_{\alpha\dot\beta} \hat{\bar\eta}^{\dot\beta} \partial_\mu \hat A - i\hat v^\mu \partial_\mu \hat\psi_\alpha - 2\hat\eta_\alpha \, \hat{\bar\eta}^{\dot\beta} \hat\chi^*_{\dot\beta} + \Omega_{\psi\,\alpha} \,. \tag{2.35}$$

By requiring the nilpotency of the BRST transformation, we find

$$s\,\hat\chi^*_{\dot\alpha} = -i\bar\sigma^\mu_{\dot\alpha\beta} \partial_\mu \hat\psi^\beta - i\hat v^\mu \partial_\mu \hat\chi^*_{\dot\alpha} + \Omega^*_{\chi,\dot\alpha} \,. \tag{2.36}$$

[4]Equations (2.33) and (2.34) can be obtained in a straightforward way by using a superspace technique: $s\,\hat X \equiv \eta^\alpha D_\alpha \hat X + v^\mu \partial_\mu \hat X + \Omega$ where X and Ω are chiral superfields and D_α is the covariant derivative. By imposing the nilpotency, $s^2 = 0$, one gets $s\,\Omega = \eta^\alpha D_\alpha \Omega + v^\mu \partial_\mu \Omega$. This means that the fields Ω transform under the supersymmetry transformations as a chiral supermultiplet.

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$\Omega^*_{\chi,\dot\alpha}$ is the shift for $\hat\chi^*_{\dot\alpha}$ and it guarantees that the cohomology is independent of the variables $\Omega^*_{\chi,\dot\alpha}$ and $\hat\chi^*_{\dot\alpha}$. The BRST transformations for $\Omega_{\psi\alpha}$ and $\Omega^*_{\chi,\dot\alpha}$ can again be derived by imposing the nilpotency of s on $\hat\psi_\alpha$ and $\hat\chi^*_{\dot\alpha}$.

These BRST transformations (2.35) and (2.36) can be implemented within the BV formalism by coupling $s\,\hat\psi_\alpha$ and $s\,\hat\chi^*_{\dot\alpha}$ to the corresponding conjugate variables ("antifields,,) $\hat\psi^*_\alpha$ and $\hat\chi_{\dot\alpha}$. We notice that in the case of $\hat\chi^*_{\dot\alpha}$ the "antifield,, is actually an external source with its own shift field $\Omega_{\chi,\alpha}$.

This example shows that we have to introduce the antifields $\hat\chi^*$ (or eventually the auxiliary fields) for each background field $\hat\phi$ on which the BRST differential squares to zero only modulo the equations of motion. This is needed in order to reproduce the correct algebra. In addition, each new antifield $\hat\chi^*$ has to be paired with a corresponding shift Ω^*_χ in order to enforce the triviality with respect to the BRST cohomology and to close the symmetry on the antifields $\hat\chi^*$.

We also remark that by using antifields instead of auxiliary fields, we loose the multiplet structure. It seems that even in the cases where the auxiliary fields cannot be found in order to establish the closure of the algebra at the level of fields, the BV technique by means of antifields is able to supply the correct content of variables to close the algebra. However, the structure of superfields is no longer available.

We can finally go back to the initial question how to define the correct splitting between the quantum and the classical fields. The master equation (2.21) has to be solved in the appropriate space of variables (including the needed auxiliary antifields)

$$\varphi^I = \{\xi^i, \xi^\alpha_C, \hat\phi^i, \hat c_\alpha, \Omega_i, \theta_\alpha, \hat\chi_i, \Omega_{\chi^i}\}\,, \qquad \varphi^*_I = \{\xi^{*,i}, \xi^{*,\alpha}_C, \hat\phi^{*,i}, \hat c^{*,\alpha}, \Omega^{*,i}, \theta^{*,\alpha}, \hat\chi^{*,i}, \Omega^{*,i}_\chi\}$$

with boundary conditions (2.27)–(2.29) and under the requirement that

$$S|_{\varphi^*_I=0,\xi^i=\xi^\alpha_C=\Omega_i=\theta_\alpha=\hat\chi_i=\Omega_{\chi^i}=0} = S_0[\hat\phi, \hat c]\,. \tag{2.37}$$

S_0 is the original gauge invariant classical action. We will discuss in the next section the problems related with the background gauge-fixing.

In the case of closed algebras no auxiliary antifields are needed and condition (2.37) is fulfilled since the implementation of the BFM yields the replacement of the original fields Φ_i, C^α with

$$\Phi_i = \hat\phi_i + \Pi_{\phi,i}\,, \qquad C^\alpha = \hat c^\alpha + \Pi_{C,\alpha}\,, \tag{2.38}$$

where $\Pi_{\phi,i}, \Pi_{C,\alpha}$ are functions of $\hat\phi_i, \hat C^\alpha, \xi_i, \xi^\alpha_C$ vanishing for $\xi_i = \xi^\alpha_C = 0$.

For open gauge algebras the implementation of the BFM requires the extension of the space of variables in which the splitting problem can be defined (due to the introduction of the auxiliary fields). Equation (2.37) then provides the relation of the full classical action S with the original classical action S_0.

The methods needed to solve this problem vary with the model at hand. Sometimes cohomological methods can be applied [31]. It may happen that a suitable choice of the generators of the original BRST differential s is enough to obtain a solution. This is the case for instance of the Topological Yang-Mills (TYM) theory, where the Jordan form of the

BRST differential [32] can be reached by a suitable field redefinition. Notice that this is only possible if one introduces the relevant set of auxiliary fields (corresponding to the twisted auxiliary fields of $N = 2$ SYM in the WZ gauge). The use of the auxiliary fields reduces the open algebra problem to a closed algebra problem. In the space of variables which includes the auxiliary fields it can be proven that the field redefinition solving the splitting problem is actually a canonical transformation. We will deal with TYM in section 3.1.

2.3 A shortcut

Sometimes, there is an easy shortcut for finding the correct splitting functions. We notice that at first order $\Pi_{\phi,i} = \xi_i$ provides a solution to equations (2.6). This suggests that if we are able to find coordinate transformations (which will eventually be expressed by means of canonical transformations [33]) such that the r.h.s. of (2.1) becomes linear in the quantum fields ξ_i, the solution $\Pi_{\phi,i} = \xi_i$ gives an all-order solution to the problem (2.6) and this allows us to identify the splitting.

Let $\Phi' = \Phi'(\Phi)$ be a suitable change of coordinates such that

$$s\,\Phi'_i = (R')_i^{\alpha}[\Phi']C_\alpha\,, \qquad s\,C_\alpha = \frac{1}{2}F_\alpha^{\beta\gamma}[\Phi(\hat{\phi},\xi)]C_\beta C_\gamma \tag{2.39}$$

and $(R')_i^{\alpha}[\Phi']$ is at most a linear function of Φ'. Then, $\Pi_{\phi,i} = \xi_i$ is the trivial solution to the splitting problem, namely $\Phi' = \hat{\phi}' + \xi_i$. Converting back to the original variables (again by means of a canonical transformation), which yield the inverted relation $\Phi = \Phi(\Phi')$, we have

$$\Phi_i = \Phi_i\left(\hat{\phi}' + \xi_i\right) = \Phi_i\left(\hat{\phi}'(\hat{\phi}) + \xi_i\right) = \hat{\phi}_i + \Pi_i(\hat{\phi},\xi)\,, \tag{2.40}$$

where we substituted $\hat{\phi}'_i = \hat{\phi}'_i(\hat{\phi})$. Notice that to invert the change of coordinates we use the theorem for implicit functions in power series.

In order to extend this analysis to more general theories, it is convenient to formulate the change of variables in the language of canonical transformations. The new set of fields and antifields are denoted by φ'^{I} and φ'^{*}_{I}; they are related to the original variables by means of the transformation rules

$$\varphi'^{I} = (\Psi,\varphi^I)' = \frac{\delta\Psi[\varphi,\varphi'^{*}]}{\delta\varphi'^{*}_{I}}\,, \qquad \varphi^*_I = (\Psi,\varphi^*_I) = \frac{\delta\Psi[\varphi,\varphi'^{*}]}{\delta\varphi^I}\,, \tag{2.41}$$

where Ψ is the generating functional of the canonical transformations. The bracket $(\cdot,\cdot)'$ is the bracket defined in (2.22) with the coordinates φ_I and φ^*_I replaced by the new variables. The transformations of the new fields are computed using again the bracket

$$s\,\varphi'_I = \left(S[\varphi',\varphi'^{*}],\varphi'_I\right)'\,, \qquad s\,\varphi'^{*}_I = (S[\varphi',\varphi'^{*}],\varphi'^{*}_I)'\,, \tag{2.42}$$

and they extend the rules given in equation (2.39). If the canonical transformations can be chosen in such a way that the new BRST transformation rules (2.42) are of the type given in equation (2.39) (with $(R')_i^{\alpha}[\varphi']$ at most a linear function of φ') or completely linear in the quantum fields (as it happens for the twisted version of $N = 2$ SYM we will analyse in section 3.1), we can split the fields and the antifields by

$$\varphi'_I = \hat{\varphi}'_I + \xi_I\,, \qquad \varphi'^*_I = \hat{\varphi}'^*_I + \xi^*_I\,, \tag{2.43}$$

where $\hat{\varphi}'_I$ and $\hat{\varphi}'^*_I$ are the background fields. The latter transform according to (2.42) where φ_I and φ^*_I are replaced by the corresponding background fields. Notice that the canonical transformations do not need to be linear and in general they are analytical functions of the fields and antifields.

Even if the background fields φ^*_I for the antifields are in principle not necessary, we found them useful as bookkeeping of the transformation rules for the background fields and they prove to be convenient in order to formulate the canonical transformation in equation (2.41).

By an other (inverse) canonical transformation, we can re-express the new variables φ'_I and φ'^*_I in terms of the older ones: this leads to the relation between the original fields φ_I and φ^*_I and the quantum fields ξ_I and ξ^*_I.

2.4 A simple example

As a warming-up example, we consider the simple topological Yang-Mills theory. This example is interesting because it displays some of the features of the $N = 2$ model that will be discussed later, but at the same time is very simple. In the present example we will show how to construct the BFM by using the Jordan form of the BRST differential, which can be reached in the present model by a simple field redefinition, and how to use the BFM to characterize the BRST cohomology and the physical observables of the theory.

The observables of the theory are not defined in terms of the BRST cohomology only, but a supplementary condition is needed. In fact, $H(s)$ and $H(s|d)$ are empty for any ghost number. This can be easily verified by using a suitable canonical transformation of variables which brings all the transformation rules into the form of trivial pairs ($s\mathcal{U} = \mathcal{V}$ and $s\mathcal{V} = 0$). This is discussed in the next paragraphs. On the other side one can define a new nilpotent BRST-like operator w associated with the gauge invariance and with the independence of the classical ungauged action from the ghost field c, such that the observables are identified by the cohomologies

$$H_{basic}(s) = \Big\{H(s|\mathcal{B})|w\mathcal{B} = 0\Big\}\,, \qquad H_{basic}(s|d) = \Big\{H(s|\mathcal{B})|w\mathcal{B} = d\mathcal{X}\Big\}\,. \tag{2.44}$$

Here $\mathcal{B}$ is the space of basic forms which are gauge-invariant and do not depend on the ghost c. In the following we will construct the BRST-like operator w and discuss the relation with the BFM.

The topological Yang-Mills theory is described by the BRST transformations:

$$s\,A = \psi - \nabla c\,, \qquad s\,\psi = [\psi, c] - \nabla\phi\,, \qquad s\,c = \phi - \frac{1}{2}[c, c]\,, \qquad s\,,\phi = [\phi, c]\,, \tag{2.45}$$

for the fields $\varphi_I = \{A, c, \psi, \phi\}$ and

$$s\,A^* \;=\; -[A^*, c] - [\psi^*, \phi]\,,$$

$$
\begin{aligned}
s\,\psi^* &= A^* + [\psi^*, c]\,, \\
s\,c^* &= \nabla A^* + [\psi^*, \psi] + [\phi^*, \phi] + [c^*, c]\,, \\
s\,\phi^* &= c^* + \nabla\psi^* + [\phi^*, c]\,,
\end{aligned}
\tag{2.46}
$$

for the antifields $\varphi^*_I = \{A^*, c^*, \psi^*, \phi^*\}$. Fields and antifields are forms with values in the Lie algebra of the underlying gauge group. The antifields φ^*_I are defined as the Hodge dual of the conventional definition, for example $A^* = A^{*\mu}\epsilon_{\mu\nu\rho\sigma}dx^\nu dx^\rho dx^\sigma$ is a 3-form in 4 dimensions. The background fields $\hat\varphi^I$ and $\hat\varphi^*_I$ transform correspondingly, according to the considerations of the previous section. The fields $\hat\varphi^*_I$ are introduced in order the quantum fields ξ_I to be coupled to their "quantum" antifields ξ^{*I}, after the splitting.

It is easy to see that with the change of coordinates $\psi' = \psi - \nabla c\,, \phi' = \phi - \frac{1}{2}[c, c]\,, A^{'*} = A^* + [\psi^*, c]$ and $c^{'*} = c^* + \nabla\psi^* + [\phi^*, c]$, generated by the functional

$$
\Psi[\varphi_I, \varphi^{'*}_I] = \int \mathrm{tr}\left[A^{'*}A + c^{'*}c + \psi^{'*}\left(\psi - \nabla c\right) + \phi^{'*}\left(\phi - \frac{1}{2}[c, c]\right)\right]\,,
\tag{2.47}
$$

the BRST transformations (2.45) and (2.46) become linear in the new variables. This leads to the consequence that the cohomologies $H(s)$ and $H(s|d)$ are empty. Then we can split the new fields in a linear way by setting

$$
\begin{aligned}
&A = \hat A + \xi_A\,, \qquad \psi' = \hat\psi' + \xi'_\psi\,, \qquad c = \hat c + \xi_c\,, \qquad \phi' = \hat\phi' + \xi'_\phi\,, \\
&A^{'*} = \hat A^{'*} + \xi'_{A^*}\,, \qquad \psi^* = \hat\psi^* + \xi_{\psi^*}\,, \qquad c^{'*} = \hat c^{'*} + \xi'_{c^*}\,, \qquad \phi^* = \hat\phi^* + \xi_{\phi^*}\,.
\end{aligned}
\tag{2.48}
$$

The BRST transformations of the new variables are the obvious ones derived from equations (2.45) and (2.46), i.e. for instance in the case of the doublet (A, ψ'): $s\hat A = \hat\psi'$, $s\,\xi_A = \xi'_\psi$, $s\hat\psi' = 0$ and $s\xi'_\psi = 0$. The old quantum fields are obtained by exploiting equation (2.45). For instance, in the case of ψ' we obtain

$$
\psi = (\hat\psi' + d\hat c + [\hat A, \hat c]) + \xi'_\psi + d\xi_c + [\hat A, \xi_c] + [\xi_A, \hat c] + [\xi_A, \xi_c]\,.
\tag{2.49}
$$

The terms in the round brackets are of order zero in the quantum fields, the remaining terms contain one or two powers of the quantum fields. Notice that the splitting in equation (2.49) is non-trivial and non-linear in the quantum fields.[5]

To define the observables of the theory, we consider again the transformation rules given in (2.45) and those for the background fields

$$
s\,\hat A = \Omega - \hat\nabla\hat c\,, \qquad s\,\Omega = [\Omega, \hat c] - \hat\nabla\theta\,, \qquad s\,\hat c = \theta - \frac{1}{2}[\hat c, \hat c]\,, \qquad s\,\theta = [\theta, \hat c]\,.
\tag{2.51}
$$

[5]In several examples, one can use a superfield notation $\mathcal{A} = c + A + F^* + \psi^* + \phi^*$ and $\mathcal{B} = \phi + \psi + F + A^* + c^*$. F is the field strength and F^* is the antifield associated with the condition of self-duality. The BRST transformations (2.45) and (2.46) can be written in a compact manner as

$$
(s + d)\,\mathcal{A} + \mathcal{A}^2 = \mathcal{B}\,, \qquad (s + d)\,\mathcal{B} + [\mathcal{A}, \mathcal{B}] = 0\,.
\tag{2.50}
$$

Thus, by introducing the new variables $\mathcal{B}' = \mathcal{B} - \mathcal{A}^2$ and $\mathcal{A}' = \mathcal{A}$, the BRST transformations are simplified and the splitting becomes trivial.

Again, it is convenient to redefine the fields $\Omega' = \Omega - \hat{\nabla}\hat{c}$ and $\theta' = \theta - \frac{1}{2}[\hat{c},\hat{c}]$ in order to simplify the relation between our notation and the one adopted in [24]. The definition of the basic forms is obtained by computing the kernel of the operator

$$\begin{aligned} w\,A &= -\nabla\hat{c}\,, \qquad & w\,\psi &= [\psi,\hat{c}]\,, \\ w\,c &= \theta' + [c,\hat{c}] \qquad & w\,\phi &= [\phi,\hat{c}]\,, \\ w\,\hat{c} &= -\frac{1}{2}[\hat{c},\hat{c}]\,, \qquad & w\,\theta' &= [\theta',\hat{c}]\,. \end{aligned} \tag{2.52}$$

As it can be readily seen the operator w is nilpotent and anticommutes with the BRST symmetry (2.45)–(2.51). This transformation rule can be extended in order to take into account the background field $\hat{A}$. We have in addition

$$w\,\hat{A} = -\hat{\nabla}\hat{c}\,, \qquad w\,\Omega' = [\Omega',\hat{c}]\,. \tag{2.53}$$

The transformation rules for the gauge field A and its partner $\hat{A}$ are the usual background gauge transformations. The rule for the background ghost $\hat{c}$ is the usual transformation for the gauge ghost. Notice that in addition the transformation for the ghost Ω' is the usual background trasformation. Finally, we have to point out that all the transformations given in (2.52) are linear in quantum fields and therefore they lead to linear WTI.

Comparing (2.52) with the operator w given in [24], one can see that all transformations do coincide except those for the background $\hat{A}$ which are indeed new. The purpose of the operator w is to restrict the space of local operators to the sector of basic forms and it is fundamental to define the observables at the quantum level. It happens that the construction of this operator in the BFM context is completely natural since the fields $\hat{c}$ and θ, necessary to implement the gauge transformations, are indeed present. We can therefore conclude that the restriction to the space of those background gauge-invariant polynomials which are independent of the ghost c (notice that this requirement is implemented by means of the linear shift into θ) gives the correct observables. The BFM is not only a useful tool to compute gauge-invariant operators correlation functions, but it is also fundamental to select the physical content of the theory.

Finally, we can summarize the results of the present section in the following remarks. We find out that, according to a cohomological analysis or by use of suitable field redefinitions, we can derive the splitting functions $\Pi_\phi, \Pi_{\phi^*}, \ldots, \Pi_{C^*}$, such that the background gauge transformations of the quantum fields ξ^I_ϕ and $\xi_{C\alpha}$ are linear in the quantum fields. By defining the operator $\mathcal{N}_{\hat{c}} = \int \hat{c}_\alpha \delta_{\hat{c}_\alpha}$, which counts the powers of the background ghost fields $\hat{c}_\alpha$, we can decompose the BRST operator s in terms of eigenvalues of $\mathcal{N}_{\hat{c}}$: $s = s_0 + s_1 + \sum_{n>1} s_n$ where s_0 represents the BRST operator for the classical BRST symmetry, s_1 entails the background gauge invariance, s_n with $n > 1$ describe the closure terms.

2.5 Background gauge fixing

The splitting problem defines a change of variables such that the new quantum fields transform linearly under the background transformations. If the splitting problem cannot be solved, the background transformations cannot be defined, independently of the perturbative quantization of the theory. In those cases where the good variables, suited for the

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implementation of the BFM, have been found, an additional problem arises: is it possible to find a suitable gauge-fixing condition compatible with the invariance of the ungauged classical action under the WT background identities? This issue can be analysed in a very general setting within the BV formalism and has already been thoroughly considered in the literature.[6] Here we only discuss some aspects relevant to the simplest case of irreducible models.

In order to construct the quantum effective action in perturbation theory, we need to compute the propagators for all quantum fields ξ_{ϕ_i} of the theory. The ungauged classical action S, fulfilling the master equation

$$(S, S) = 0\,, \tag{2.54}$$

gives rise to a matrix of 2-point functions which is in general non-invertible. In order to remove this degeneracy, S must be modified by adding non-minimal sectors. Then one performs a canonical transformation, generated by the gauge-fixing fermion $\Psi_{g.f.}$, in such a way that the transformed classical action yields well-defined propagators for all quantum fields.

The ungauged classical action S depends on the background fields $\hat{\phi}_i$ and on the new quantum fields ξ_{ϕ_i}. It fulfills the background WT identities, under which ξ_{ϕ_i} transform linearly. The addition of the non-minimal sectors, needed to fix the gauge, and the canonical transformation generated by the gauge-fixing fermion $\Psi_{g.f.}$ should not break this WT invariance.

The minimal sectors we will analyse involve one generation of antighost fields $\bar{c}^\alpha$ and Lagrange multipliers B^α, together with the corresponding antifields. The index α runs over those fields ϕ_α whose 2-point function matrix $\{\Gamma_{\phi_\alpha \phi_\beta} = \frac{\delta^2 S}{\delta\phi_\alpha \delta\phi_\beta}\}$, computed from the ungauged classical action S, is not invertible.

As a first step, we add to the ungauged classical action S the non-minimal terms

$$S_{\text{n.m.}} = \int \bar{c}^{*\alpha} B_\alpha\,. \tag{2.55}$$

Then we implement the gauge-fixing by means of a canonical transformation generated by the gauge-fixing fermion functional

$$\Psi_{g.f.}[\hat{\phi}, \xi] = \int \bar{c}^\alpha \mathcal{F}_\alpha(\hat{\phi}_i, \xi_{\phi_i}, \hat{c}^\alpha, \xi_C\,; B^\alpha)\,. \tag{2.56}$$

Let us denote by $\Gamma^{(0)}$ the action obtained from $S + S_{n.m.}$ after the gauge-fixing canonical transformation has been performed. The $\mathcal{F}_\alpha$ in equation (2.56) are chosen in such a way that, after the canonical transformation, the complete matrix of the 2-point functions, computed from $\Gamma^{(0)}$, is invertible.

The extension of the background transformations to the generators $(\bar{c}^\alpha, B^\alpha)$ of the non-minimal sector must yield background transformations for $(\bar{c}^\alpha, B^\alpha)$ which are linear in the quantum fields. Moreover, we also require that the transformed gauge-fixed classical action $\Gamma^{(0)}$ obeys the background WT invariance. This requirement is fulfilled if we impose that the functional $\Psi_{g.f.}$ is background-gauge invariant:

$$\delta_{bkg} \Psi_{g.f.} = 0\,, \tag{2.57}$$

[6] For a review see e.g. ref. [34].

where δ_{bkg} denotes here the component of s of degree one in the background ghost fields (the generator of the background transformations), properly extended to the non-minimal sector. From equation (2.57) one gets

$$\delta_{bkg}\bar{c}^{\alpha}\mathcal{F}^{\alpha} - \bar{c}^{\alpha}\delta_{bkg}\mathcal{F}^{\alpha} = 0\,. \tag{2.58}$$

By taking into account the above equation and the requirement of the linearity of $\delta_{bkg}\bar{c}^{\alpha}$ we obtain

$$\delta_{bkg}\bar{c}^{\alpha} = \mathcal{M}^{\alpha\beta}(\hat{\phi},\hat{c})\bar{c}^{\beta}\,, \tag{2.59}$$

where $\mathcal{M}^{\alpha\beta}(\hat{\phi},\hat{c})$ is independent of the quantum fields $\xi_{\phi_i}, \bar{c}^{\alpha}, B^{a}$. Eq. (2.59) provides the natural definition for the background transformation of B^{α}:

$$\delta_{bkg}B^{\alpha} = \mathcal{M}^{\alpha\beta}(\hat{\phi},\hat{c})B^{\beta}\,. \tag{2.60}$$

By substituting eq. (2.59) into eq. (2.58) we get that the functions $\mathcal{F}_{\alpha}(\hat{\phi}_i, \xi_{\phi_i}, \hat{c}^{\alpha}, \xi_c\,; B^{\alpha})$ should transform as follows under δ:

$$\delta_{bkg}\mathcal{F}^{\alpha} = \mathcal{M}^{\beta\alpha}\mathcal{F}_{\beta}\,. \tag{2.61}$$

The fact that the fields ξ_{ϕ_i} and ξ_C transform under linear background gauge transformations simplifies the construction of the functions $\mathcal{F}_{\alpha}$: it turns out that in many cases, as for instance in ordinary gauge theories, they can be obtained from their background-independent component by covariantizing the differential operators with respect to the background fields.

The case of TYM, which we will analyse in section 3.1, is rather peculiar. There we first perform the gauge-fixing of the classical action in terms of the original unsplitted variables. The solution to the splitting problem yields for TYM a set of variables that transform linearly under the full BRST differential, which can hence be identified with the generator of the background symmetry. As a consequence, the gauge-fixing term does not need to be modified to respect the background invariance.

Once a background covariant gauge-fixing has been introduced, the quantum effective action can be constructed in perturbation theory. The symmetry requirements of background invariance and ST invariance at the quantum level can be discussed along the lines of [26, 11, 2, 3].

As a final point, we would like to emphasize that the BFM construction of physical connected amplitudes requires the introduction of an additional gauge-fixing term for the classical background gauge fields [2]. The latter does not affect the computation of the quantum effective action and only enters in the BFM computation of connected amplitudes of BRST-invariant local operators. A complete discussion of the interplay between this background gauge-fixing term, the background WT identities and the ST identities is provided in [2, 3].

3. $N=2$ super Yang-Mills

3.1 Topological Yang-Mills theory

In this section we show how the background field method can be implemented for $N=2$ super-Yang-Mills in the Wess-Zumino gauge. We will work within the flat euclidean space-time. In order to construct the correct splitting of the fields into a background and a quantum part, with the latter transforming linearly under the background symmetry, we consider the off-shell formulation of the supersymmetry algebra of twisted $N=2$ super-Yang-Mills in the Wess-Zumino (WZ) gauge.

In the off-shell formulation the fields of $N=2$ super-Yang-Mills in the WZ gauge consist of a gauge field A_μ, two spinors ψ^i_α, $i=1,2$ and the conjugate $\bar{\psi}^i_{\dot\alpha}$, two scalars $\phi, \bar\phi$ ($\bar\phi$ being the complex conjugate of ϕ) and an SU(2) triplet of auxiliary fields $b^{ij}=b^{ji}$, $i,j=1,2$.

After the twisting and the identification of the interal index i with the spinor index α, the spinor $\bar\psi^i_{\dot\alpha}$ can be related to an anticommuting vector ψ_μ given by

$$\psi_\mu = (\bar\sigma_\mu)^{\dot\alpha\alpha}\bar\psi_{\alpha\dot\alpha}\,. \tag{3.1}$$

The fields $\psi_{\alpha\beta}$ are decomposed into their symmetric component $\psi_{(\alpha\beta)}$ and their antisymmetric component $\psi_{[\alpha\beta]}$:

$$\psi_{\alpha\beta} = \psi_{(\alpha\beta)} + \psi_{[\alpha\beta]}\,. \tag{3.2}$$

$\psi_{(\alpha\beta)}$ is related to an antisymmetric self-dual anticommuting field $\chi_{\mu\nu}$ via the definition

$$\chi_{\mu\nu} = \tilde\chi_{\mu\nu} = (\sigma_{\mu\nu})^{\alpha\beta}\psi_{(\alpha\beta)} \tag{3.3}$$

where $\tilde\chi_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\chi^{\rho\sigma}$.

The antisymmetric component $\psi_{[\alpha\beta]}$ is associated to the anticommuting scalar η given by

$$\eta = \epsilon^{\alpha\beta}\psi_{[\alpha\beta]}\,. \tag{3.4}$$

Finally the auxiliary fields $b_{\alpha\beta}$ are related to the antisymmetric commuting self-dual field $b_{\mu\nu}$ defined by

$$b_{\mu\nu} = (\sigma_{\mu\nu})^{\alpha\beta}b_{\alpha\beta}\,. \tag{3.5}$$

Therefore the off-shell multiplet of $N=2$ super-Yang-Mills in the Wess-Zumino gauge $(A_\mu, \psi^i_\alpha, \bar\psi^i_{\dot\alpha}, \phi, \bar\phi, b^{ij})$ is transformed into the twisted multiplet

$$(A_\mu, \psi_\mu, \chi_{\mu\nu}, \eta, \phi, \bar\phi, b_{\mu\nu})\,,$$

providing the field content of topological Yang-Mills theory (TYM) in the off-shell formulation.

The classical action of TYM is given by

$$S_{TYM} = \frac{1}{g^2}\,\mathrm{Tr}\int d^4x\Bigg(+\frac{1}{2}F^{-}_{\mu\nu}F^{\mu\nu -} - \frac{1}{2}b_{\mu\nu}b^{\mu\nu} - \chi^{\mu\nu}(D_\mu\psi_\nu - D_\nu\psi_\mu)^{-} + \\ + \eta D_\mu\psi^\mu - \frac{1}{2}\bar\phi D_\mu D^\mu\phi + \frac{1}{2}\bar\phi\{\psi^\mu,\psi_\mu\} - \\ - \frac{1}{2}\phi\,\{\chi^{\mu\nu},\chi_{\mu\nu}\} - \frac{1}{8}[\phi,\eta]\eta - \frac{1}{32}[\phi,\bar\phi][\phi,\bar\phi]\Bigg)\,. \tag{3.6}$$

The action in equation (3.6) coincides with the one given in [20] when the equation of motion for the auxiliary field $b_{\mu\nu}$ is imposed. D_μ is the covariant derivative given by $D_\mu(\cdot) = \partial_\mu\cdot + [A_\mu,\cdot]$. We denote by a $-$ the self-dual component of a tensor, so that

$$F^{-}_{\mu\nu} = F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,. \tag{3.7}$$

$F^{-}_{\mu\nu}$ fulfills

$$\tilde F^{-}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma -} = F^{-}_{\mu\nu}\,. \tag{3.8}$$

As is well-known [20], the classical TYM action can be regarded as the twisted version of $N = 2$ super-Yang-Mills theory in the Wess-Zumino gauge. As a consequence, in addition to gauge invariance, the classical TYM action exhibits further symmetries generated by the twisted $N = 2$ supersymmetry generators [20]. The set of these generators contains a scalar generator δ, a vector generator δ_μ and a self-dual tensor generator $\delta_{\mu\nu}$, where δ is to be identified with Witten's fermionic symmetry [35].

For our purposes we find it convenient to gather the BRST symmetry s, issued from gauge invariance of S_{TYM} in equation (3.6), the scalar symmetry δ and the vector symmetry δ_μ, together with translation invariance, into a single BRST differential $\mathcal{Q}$ [20], given by

$$\mathcal{Q} = s + \omega\delta + \epsilon^\mu\delta_\mu + v^\mu\partial_\mu - \epsilon^\mu\frac{\partial}{\partial v^\mu}\,. \tag{3.9}$$

Here, ω is a commuting constant external source associated with Witten's fermionic symmetry. We remark that, unlike in [20], ω does not carry any ghost number. This is reflected in our assignment of the ghost number for the fields of the model: ψ_μ is assumed to carry ghost number 1, ϕ ghost number 2, $\chi_{\mu\nu}$ and η ghost number -1, while $\bar\phi$ carries ghost number -2; A_μ and $b_{\mu\nu}$ carry zero ghost number. The constant external source associated with the vector symmetry is denoted by ϵ^μ, the constant external source associated with translations by v^μ. With our assignments, ϵ_μ carries ghost number 2, while v^μ carries ghost number 1.

As noted in [20], we can discard the tensor generator $\delta_{\mu\nu}$, since it does not carry additional information with respect to the subalgebra generated by s, δ, δ_μ and ∂_μ. The explicit form of the operator $\mathcal{Q}$ is given in appendix A. Since we are using the off-shell formalism with the auxiliary fields $b_{\mu\nu}$, $\mathcal{Q}^2 = 0$. In the on-shell formalism adopted e.g. in [20], where the auxiliary fields $b_{\mu\nu}$ are eliminated via their equation of motion, the operator $\mathcal{Q}$ is nilpotent only modulo the equations of motions of ψ_μ and $\chi_{\mu\nu}$; the corresponding STI can be written by adding suitable terms quadratic in the antifields coupled to ψ_μ and $\chi_{\mu\nu}$.

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We can now gauge-fix the classical TYM action by choosing [20]:

$$\begin{aligned} S_{gf} &= \mathcal{Q}\int d^4x \,\mathrm{Tr}\,(\bar{c}\partial A) \\ &= \mathrm{Tr}\int d^4x \left(b\partial A + \bar{c}\partial^\mu D_\mu c - \omega\bar{c}\partial^\mu\psi_\mu - \frac{\epsilon^\nu}{2}\bar{c}\partial^\mu\chi_{\nu\mu} - \frac{\epsilon^\mu}{8}\bar{c}\partial_\mu\eta \right). \end{aligned} \tag{3.10}$$

In the above equation $\bar{c}$ is the antighost field and b is the Nakanishi-Lautrup multiplier field. The gauge-fixed classical action

$$\Sigma = S_{TYM} + S_{gf} \tag{3.11}$$

is $\mathcal{Q}$-invariant.

In order to write the ST identities we couple the $\mathcal{Q}$-variations of the fields to the corresponding antifields as follows:

$$\begin{aligned} S_{\mathrm{ext}} = \mathrm{Tr}\int d^4x \Big(& c^*\mathcal{Q}c + \phi^*\mathcal{Q}\phi + A^{\mu *}\mathcal{Q}A_\mu + \psi^{\mu *}\mathcal{Q}\psi_\mu + \bar{c}^*\mathcal{Q}\bar{c} + \\ & + b^*\mathcal{Q}b + \bar{\phi}^*\mathcal{Q}\bar{\phi} + \eta^*\mathcal{Q}\eta + \frac{1}{2}\chi^{\mu\nu *}\mathcal{Q}\chi_{\mu\nu} + \frac{1}{2}b^{\mu\nu *}\mathcal{Q}b_{\mu\nu}\Big) + v^{\mu *}\mathcal{Q}v_\mu \,. \end{aligned} \tag{3.12}$$

The full classical action is then given by

$$\Gamma^{(0)} = S_{TYM} + S_{gf} + S_{ext} \tag{3.13}$$

and fulfills the following ST identities:

$$\begin{aligned} \mathcal{S}(\Gamma^{(0)}) = \mathrm{Tr}\int d^4x \Bigg(& \frac{\delta\Gamma^{(0)}}{\delta A^\mu}\frac{\delta\Gamma^{(0)}}{\delta A^{\mu *}} + \frac{\delta\Gamma^{(0)}}{\delta\psi^{\mu *}}\frac{\delta\Gamma^{(0)}}{\delta\psi^\mu} + \frac{\delta\Gamma^{(0)}}{\delta c^*}\frac{\delta\Gamma^{(0)}}{\delta c} + \frac{\delta\Gamma^{(0)}}{\delta\phi^*}\frac{\delta\Gamma^{(0)}}{\delta\phi} + \frac{\delta\Gamma^{(0)}}{\delta\bar{\phi}^*}\frac{\delta\Gamma^{(0)}}{\delta\bar{\phi}} + \\ & + \frac{\delta\Gamma^{(0)}}{\delta\eta^*}\frac{\delta\Gamma^{(0)}}{\delta\eta} + \frac{1}{2}\frac{\delta\Gamma^{(0)}}{\delta\chi^{\mu\nu *}}\frac{\delta\Gamma^{(0)}}{\delta\chi_{\mu\nu}} + \frac{1}{2}\frac{\delta\Gamma^{(0)}}{\delta b^{\mu\nu *}}\frac{\delta\Gamma^{(0)}}{\delta b_{\mu\nu}} + \frac{\delta\Gamma^{(0)}}{\delta\bar{c}^*}\frac{\delta\Gamma^{(0)}}{\delta\bar{c}} + \\ & + \frac{\delta\Gamma^{(0)}}{\delta b^*}\frac{\delta\Gamma^{(0)}}{\delta b}\Bigg) + \frac{\delta\Gamma^{(0)}}{\delta v^*_\mu}\frac{\delta\Gamma^{(0)}}{\delta v^\mu} \\ = \frac{1}{2}\left(\Gamma^{(0)},\Gamma^{(0)}\right) = 0\,, & \end{aligned} \tag{3.14}$$

where the bracket is defined as

$$(X,Y) = \int d^4x \sum_I \sigma_I \left(\frac{\delta X}{\delta\Phi_I}\frac{\delta Y}{\delta\Phi^*_I} - (-1)^{(\epsilon_X+1)}\frac{\delta X}{\delta\Phi^*_I}\frac{\delta Y}{\delta\Phi_I} \right). \tag{3.15}$$

In the above equation, $\Phi_I = \{A_\mu, \psi_\mu, \chi_{\mu\nu}, b_{\mu\nu}, \eta, \bar{\phi}, c, \phi, \bar{c}, b, v_\mu\}$ and $\sigma_I = 0$ for all fields but $\chi_{\mu\nu}, b_{\mu\nu}$, for which $\sigma_{\chi_{\mu\nu}} = \sigma_{b_{\mu\nu}} = 1/2$. This factor is needed to take into account antisymmetry in the Lorentz indices of $\chi_{\mu\nu}, b_{\mu\nu}$. the term $\epsilon(X)$ stands for the statistics of X ($\epsilon(X) = 0$ if X is a boson, $\epsilon(X) = 1$ is X is a fermion).

We also introduce the linearized ST operator $\tilde{\mathcal{Q}}$, given by

$$\tilde{\mathcal{Q}} = \left(\Gamma^{(0)}, \cdot\right). \tag{3.16}$$

Now we redefine the fields as follows:

$$
\begin{aligned}
\omega\psi'_\mu - \partial_\mu c \equiv \tilde{\mathcal{Q}} A_\mu &= \omega\psi_\mu - \partial_\mu c + \cdots\,,\\
\omega b'_{\sigma\tau} \equiv \tilde{\mathcal{Q}}\chi_{\sigma\tau} &= \omega b_{\sigma\tau} + \cdots\,,\\
2\omega\eta' = \tilde{\mathcal{Q}}\bar{\phi} &= 2\omega\eta + \cdots\,,\\
-\omega^2\phi' = \tilde{\mathcal{Q}} c &= -\omega^2\phi + \cdots\,,\\
b' = \tilde{\mathcal{Q}}\bar{c} &= b + v^\mu\partial_\mu\bar{c}\,,
\end{aligned}
\tag{3.17}
$$

while we leave all other fields unchanged. Notice that this transformation is invertible. Apart from the fields (A_μ, ψ'_μ) only $\tilde{\mathcal{Q}}$-doublets are now present. Notice that the transformation generated by $\tilde{\mathcal{Q}}$ in equation (3.17) is now linear in the new quantum fields.

Explicitly we get:

$$
\begin{aligned}
\psi'_\mu &= \psi_\mu - \frac{1}{\omega}[A_\mu, c] + \frac{\epsilon^\nu}{2\omega}\chi_{\nu\mu} + \frac{\epsilon_\mu}{8\omega}\eta + \frac{v^\nu}{\omega}\partial_\nu A_\mu\,,\\
b'_{\sigma\tau} &= b_{\sigma\tau} + \frac{1}{\omega}\{c, \chi_{\sigma\tau}\} + F^-_{\sigma\tau} + \frac{\epsilon^\mu}{8\omega}(\epsilon_{\mu\sigma\tau\nu} + g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma})D^\nu\bar{\phi} + \frac{v^\nu}{\omega}\partial_\nu\chi_{\sigma\tau}\,,\\
\eta' &= \eta + \frac{1}{2\omega}[c, \bar{\phi}] + \frac{1}{2\omega}v^\nu\partial_\nu\bar{\phi}\,,\\
\phi' &= \phi - \frac{1}{\omega^2}c^2 + \frac{\epsilon^\mu}{\omega^2}A_\mu - \frac{\epsilon^2}{16\omega^2}\bar{\phi} + \frac{v^\nu}{\omega^2}\partial_\nu c\,,\\
b' &= b + v^\mu\partial_\mu\bar{c}\,.
\end{aligned}
\tag{3.18}
$$

The role played by ϵ_μ and v_μ is rather suggestive: they can be thought as background fields entering into the field redefinition. From the cohomological point of view this is confirmed by the fact that (v^μ, ϵ^μ) form a set of doublets under $\tilde{\mathcal{Q}}$. We remark that the field redefinition in equation (3.17) gives terms that are not analytic in ω. We will discuss in section 3.2 how the BFM allows naturally to recover the observables of the model by taking into account the relevant equivariant cohomology of TYM.

At this point we can perform a linear splitting in the primed variables

$$
\psi'_\mu = \hat{\psi}_\mu + \xi_{\psi_\mu}\,, \qquad b'_{\sigma\tau} = \hat{b}_{\sigma\tau} + \xi_{b_{\sigma\tau}}\,, \qquad \eta' = \hat{\eta} + \xi_\eta\,, \qquad \phi' = \hat{\phi} + \xi_\phi\,, \qquad b' = \hat{b} + \xi_b\,,
\tag{3.19}
$$

and then go back to reconstruct the full non-linear splitting, making use of equation (3.18). Notice that also the fields that are unchanged under the field redefinition in equation (3.17) are understood to be splitted into a background and a quantum part:

$$
\begin{aligned}
A_\mu &= \hat{A}_\mu + \xi_{A,\mu}\,, \qquad \chi_{\sigma\tau} = \hat{\chi}_{\sigma\tau} + \xi_{\chi,\sigma\tau}\,, \qquad \bar{\phi} = \hat{\bar{\phi}} + \xi_{\bar{\phi}}\,,\\
c &= \hat{c} + \xi_c\,, \qquad \bar{c} = \hat{\bar{c}} + \xi_{\bar{c}}\,.
\end{aligned}
\tag{3.20}
$$

The corresponding BRST transformations of the new variables are given by

$$
\begin{aligned}
\tilde{\mathcal{Q}}\hat{A}_\mu &= \omega\hat{\psi}_\mu - \partial_\mu\hat{c}\,, \qquad \tilde{\mathcal{Q}}\hat{\psi}_\mu = -\omega\partial_\mu\hat{\phi}\,,\\
\tilde{\mathcal{Q}}\xi_{A,\mu} &= \omega\xi_{\psi_\mu} - \partial_\mu\xi_c\,, \qquad \tilde{\mathcal{Q}}\xi_{\psi_\mu} = -\omega\partial_\mu\xi_\phi\,,
\end{aligned}
\tag{3.21}
$$

$$
\tilde{\mathcal{Q}}\hat{\chi}_{\sigma\tau} = \hat{b}_{\sigma\tau}\,, \qquad \tilde{\mathcal{Q}}\hat{b}_{\sigma\tau} = 0\,, \qquad \tilde{\mathcal{Q}}\xi_{\chi,\sigma\tau} = \xi_{b,\sigma\tau}\,, \qquad \tilde{\mathcal{Q}}\xi_{b,\sigma\tau} = 0\,,
\tag{3.22}
$$

and analogously for the other sets of $\tilde{\mathcal{Q}}$-doublets.

As an example, in the case of ψ_μ we get

$$\begin{aligned}
\psi_\mu = {} & \hat{\psi}_\mu + \xi_{\psi_\mu} + \frac{1}{\omega}[\hat{A}_\mu, \hat{c}] + \frac{1}{\omega}[\hat{A}_\mu, \xi_c] + \frac{1}{\omega}[\xi_{A,\mu}, \hat{c}] + \frac{1}{\omega}[\xi_{A,\mu}, \xi_c] - \\
& - \frac{\epsilon^\nu}{2\omega}(\hat{\chi}_{\nu\mu} + \xi_{\chi,\mu\nu}) - \frac{v^\nu}{\omega}\partial_\nu(\hat{A}_\mu + \xi_{A,\mu}) - \\
& - \frac{\epsilon_\mu}{8\omega}\Big(\bar{\eta} + \xi_\eta - \frac{1}{2\omega}[\hat{c}, \hat{\bar{\phi}}] - \frac{1}{2\omega}[\hat{c}, \xi_{\bar{\phi}}] - \\
& \qquad - \frac{1}{2\omega}[\xi_c, \hat{\bar{\phi}}] - \frac{1}{2\omega}[\xi_c, \xi_{\bar{\phi}}] - \frac{1}{2\omega}v^\nu\partial_\nu(\hat{\bar{\phi}} + \xi_{\bar{\phi}})\Big) . \qquad (3.23)
\end{aligned}$$

Note that this splitting contains terms that are non-linear in the quantum fields.

We recover the original $\tilde{\mathcal{Q}}$ transformation of ψ_μ by acting with the $\tilde{\mathcal{Q}}$ transformations in equation (3.21) on the r.h.s. of equation (3.23).

We remark that, since the change of variables in equation (3.17) only involves fields, it is automatically a canonical transformation in the space spanned by the fields and the antifields of the model. This is analogous to the previous example of TYM. Therefore we do not modify the cohomology of the model while implementing the background splitting. In section 3.2 we show how to recover the relevant equivariant cohomology from the BFM.

A comment on the gauge-fixing function for TYM is in order. In the case of TYM we have been able to prove that the original BRST symmetry can be linearized by a suitable change of variables. This change of variables can be implemented via a canonical transformation, thus leaving the cohomology invariant. As a consequence, the classical ST identities in equation (3.14) hold. Moreover, they are already linear in the quantum fields, when expressed in terms of the new variables. We remark that these identities are fulfilled by the classical action whose gauge-fixing condition is the one given in equation (3.10). No special choice of the gauge-fixing function is needed in the present case, since the full BRST symmetry becomes linear. This should be compared with the different situation in ordinary Yang-Mills theory (see section 2.1). In these case the full BRST transformation cannot be cast in a linear form (see equation (2.11)), but one can establish an additional background WT identity, provided that a suitable background-dependent choice of the gauge-fixing function is made, as discussed in section 2.5.

With the conventions of [20] on supersymmetry in euclidean space-time we can go back to the original model $N = 2$ SYM by using the map

$$\begin{aligned}
\psi_{(\alpha\beta)} &= \frac{1}{4}(\sigma^{\mu\nu})_{\alpha\beta}\chi_{\mu\nu} , \qquad & \psi_{[\alpha\beta]} &= \frac{1}{2}\epsilon_{\alpha\beta}\eta , \\
\bar{\psi}_{\alpha\dot{\alpha}} &= -\frac{1}{2}(\sigma^\mu)_{\alpha\dot{\alpha}}\psi_\mu , \qquad & b_{\alpha\beta} &= \frac{1}{4}(\sigma^{\mu\nu})_{\alpha\beta}b_{\mu\nu} , \qquad (3.24)
\end{aligned}$$

while the fields $A_\mu, \phi, \bar{\phi}$ are mapped into themselves. Due to the linearity of this map, the correct background splitting of the original $N = 2$ multiplet in the WZ gauge can be directly recovered from the splitting of the twisted multiplet of TYM.

3.2 Equivariant cohomology for TYM and the BFM

We showed that in order to introduce the BFM for $N = 2$ SYM, it is convenient to perform the twisting of the fields and to rewrite the theory as a topological model. By the canonical change of variables in equation (3.17) the BRST operator $\mathcal{Q}$ can be cast in the form $\mathcal{Q} = \int d^4x \mathcal{V} \delta/\delta\mathcal{U}$, hence the cohomology of $\mathcal{Q}$ in the space of local formal power series vanishes. In addition, we know that $N = 2$ SYM has a physical set of observables whose correlation functions do not vanish. Hence the observables of the theory should be defined not as the BRST cohomology on the entire space, but the latter should be restricted to a suitable subspace.

Following [24], the correct set of observables is given by the BRST cohomology computed in the space of gauge invariant polynomials which are independent of the gauge ghost (in the literature this space is denoted as the space of *basic forms*). Within the BFM the observables are defined as the *BRST cohomology computed in the space of background gauge invariant polynomials which are independent of the gauge ghost* c. This suggests that there exists a new nilpotent BRST operator w, associated with the background gauge symmetry (as in the example in section 2.4), which permits to select the space of basic forms.

As has been discussed in the previous section, we have to give up the analyticity in the ghost ω in order to implement consistently the splitting and the background invariance (with respect to the full BRST transformation generated by $\mathcal{Q}$) of the theory. However, the analiticity in ω turned to be a crucial ingredient in the analysis performed in [20]. There, it has been shown how the request of analyticity allows to select the correct equivariant cohomology. Moreover, it can be proven that by introducing a suitable grading of the fields in the theory the equivariant cohomology can always be selected by the space of polynomials in ω. We cannot impose the analyticity requirement in order to identify the correct subspace, but we can construct a new differential w whose kernel identifies the basic forms. The differential w is associated to the background gauge symmetry of the theory, therefore we neglect for the moment the contributions due to the supersymmetry and translations by setting $\epsilon^\mu = v^\mu = 0$. We will also drop the non-minimal doublets $(\bar{c}, b)$ since they are cohomologically trivial.

The powers of ω entering the $\mathcal{Q}$-transformations induce a grading under which we can decompose $\mathcal{Q}$ as a sum of three terms:

$$\mathcal{Q} = \tilde{\mathcal{Q}}_{-1} + \omega \tilde{\mathcal{Q}}_0 + \omega^2 \tilde{\mathcal{Q}}_1 \,. \tag{3.25}$$

Explicitly, we have

$$\begin{aligned}
\tilde{\mathcal{Q}}_0 A_\mu &= \psi_\mu \,, \qquad \tilde{\mathcal{Q}}_0 \psi_\mu = -D_\mu \phi \,, \\
\tilde{\mathcal{Q}}_0 \chi_{\sigma\tau} &= F^-_{\sigma\tau} + b_{\sigma\tau} \,, \\
\tilde{\mathcal{Q}}_0 b_{\mu\nu} &= -(D_\mu \psi_\nu - D_\nu \psi_\mu)^- + [\chi_{\mu\nu}, \phi] \,, \\
\tilde{\mathcal{Q}}_0 \eta &= \frac{1}{2}[\phi, \bar{\phi}] \,, \qquad \tilde{\mathcal{Q}}_0 \phi = 0 \,, \\
\tilde{\mathcal{Q}}_0 \bar{\phi} &= 2\eta \,, \qquad \tilde{\mathcal{Q}}_0 c = 0 \,.
\end{aligned} \tag{3.26}$$

It is convenient to introduce $\tilde{b}_{\sigma\tau} = F^-_{\sigma\tau} + b_{\sigma\tau}$ and $\bar{\phi} \to \frac{\bar{\phi}}{2}$ to simplify these transformation rules. Now it is clear that $\tilde{\mathcal{Q}}_0$ is the de Rham operator and $(\tilde{\mathcal{Q}}_0)^2 = \mathcal{L}_\phi$ where $\mathcal{L}_\phi$ is the Lie derivative generated by the field ϕ. The cohomology that we are looking for is the de

Rham cohomology on the space of polynomials which are gauge invariant and independent of the ghost c. This means that the operator w we are looking for is given by

$$\begin{aligned} wA_\mu &= -D_\mu \hat{c}\,, \qquad w\psi_\mu = \{\hat{c}, \psi_\mu\}\,, \\ w\chi_{\sigma\tau} &= \{\hat{c}, \chi_{\sigma\tau}\}\,, \qquad w\tilde{b}_{\mu\nu} = [\hat{c}, \tilde{b}_{\mu\nu}]\,, \\ w\eta &= \{\hat{c}, \eta\}\,, \qquad w\phi = [\hat{c}, \phi]\,, \\ w\bar{\phi} &= [\hat{c}, \bar{\phi}]\,, \qquad wc = \{\hat{c}, c\} - \hat{\phi}\,, \\ w\hat{c} &= \hat{c}^2\,, \qquad w\hat{\phi} = [\hat{c}, \hat{\phi}]\,, \end{aligned} \tag{3.27}$$

The fields $\hat{c}$ and $\hat{\phi}$ correspond to the background ghost field and to the background of ϕ respectively. Their BRST transformation are the usual contractible pair transformation

$$\tilde{\mathcal{Q}}_1 \hat{c} = -\hat{\phi}\,, \qquad \tilde{\mathcal{Q}}_1 \hat{\phi} = 0\,. \tag{3.28}$$

As explained in the previous section, we also have all the background fields present. They transform under the BRST symmetry in the standard way, and under w as a gauge transformation generated by $\hat{c}$:

$$\begin{aligned} w\hat{A}_\mu &= -\hat{D}_\mu \hat{c}\,, \qquad w\hat{\psi}_\mu = \{\hat{c}, \hat{\psi}_\mu\}\,, \\ w\hat{\chi}_{\sigma\tau} &= \{\hat{c}, \hat{\chi}_{\sigma\tau}\}\,, \qquad w\hat{\tilde{b}}_{\mu\nu} = [\hat{c}, \hat{\tilde{b}}_{\mu\nu}]\,, \\ w\hat{\eta} &= \{\hat{c}, \hat{\eta}\}\,, \qquad w\hat{\bar{\phi}} = [\hat{c}, \hat{\bar{\phi}}]\,. \end{aligned} \tag{3.29}$$

The operator w generates the gauge transformations of the model. Being linear in the quantum fields, it is possible to write linear WTI. We notice that by imposing the BRST invariance, and being all the background fields cohomologically trivial, it turns out that the observables will depend only upon the original fields. Finally, the background gauge invariance, as expressed by equations. (3.27) and (3.29), selects the gauge invariant observables. The linear shift of the background gauge transformation of c implies that a gauge invariant operator annihilated by w is independent of c. According to [24], the basic forms are identified with those local integrated functionals belonging to the kernel of w.

The analysis of the cohomology follows the discussion of [23]. We only point out that the generators of the equivariant cohomology classes are given by the polynomials

$$P(\phi) = \sum_n c_n \,\mathrm{Tr}(\hat{\phi} + \xi_\phi)^{2n} \tag{3.30}$$

where c_n are numeric coefficients and the field ϕ is split into the quantum and background part. The combination $\hat{\phi} + \xi_\phi$ is fixed by the BRST symmetry and by the background gauge symmetry.

In addition, we have that these polynomials are not BRST exact, indeed if they were they would have the form

$$P(\phi) = \sum_n c_n \mathcal{Q}\left(\mathrm{Tr}(\hat{c} + \xi_c)(\hat{\phi} + \xi_\phi)^{2n-1} + \cdots\right)\,, \tag{3.31}$$

but $\mathrm{Tr}(\hat{c} + \xi_c)(\hat{\phi} + \xi_\phi)^{2n-1} + \cdots$ do not belong to the kernel of w since $\hat{c}$ transforms into $\hat{\phi}$.

Following the discussion in [23], there are further cohomological classes that are not eliminated by the previous argument, based on the background gauge invariance, for instance the operator:

$$\Delta_{\mu\nu} = \left(F^-_{\mu\nu} + b_{\mu\nu}\right)\phi\,. \tag{3.32}$$

This operator is not related to any observables of the $N = 2$ SYM theory and therefore it should be absent from the cohomology. In [23] it is excluded by imposing the supersymmetry with ϵ^μ. In our framework, by taking into account the complete background symmetry for the complete differential $\mathcal{Q}$ instead of its gauge part, we found that the invariance under $\mathcal{Q}$ (which now contains the supersymmetry in its twisted version) excludes $\Delta_{\mu\nu}$.

4. Conclusions and outlook

We have discussed the implementation of the background field method from the geometrical point of view. Within the BV formalism we have shown how the method should be generalized so as to deal with open algebras and field-dependent structure constants. This requires the identification of the proper space of variables on which the BFM splitting problem can be defined. In addition, we have underlined the similarity between the background field method and the BRST symmetry for topological gauge theories. Using this idea, we have been able to formulate the BFM for $N = 2$ SYM by introducing a field redefinition that brings the model in its topological twisted version. Therefore, the required field splitting can be implemented by a canonical transformation. We have analysed the compatibility of this field redefinition with the gauge-fixing procedure. Finally we have shown that the BRST symmetry plus the background symmetry (which is now extended to all the symmetries of the model) lead to the correct equivariant cohomology, needed to define the proper set of observables in $N = 2$ SYM. These results should be regarded as a step towards the construction of a super BFM for the MSSM.

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A. BRST transformations for TYM

The operator $\mathcal{Q}$ in equation (3.9) acts as follows on the fields of TYM:

$$\begin{aligned}
\mathcal{Q}A_\mu &= -D_\mu c + \omega\psi_\mu + \frac{\epsilon^\nu}{2}\chi_{\nu\mu} + \frac{\epsilon_\mu}{8}\eta + v^\nu\partial_\nu A_\mu\,,\\
\mathcal{Q}\psi_\mu &= \{c,\psi_\mu\} - \omega D_\mu\phi + \epsilon^\nu\left(F_{\nu\mu} - \frac{1}{2}F^-_{\nu\mu} - \frac{1}{4}b_{\nu\mu}\right) - \frac{\epsilon_\mu}{16}[\phi,\bar\phi] + v^\nu\partial_\nu\psi_\mu\,,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{Q}\chi_{\sigma\tau} &= \{c,\chi_{\sigma\tau}\} + \omega F^{-}_{\sigma\tau} + \omega b_{\sigma\tau} + \frac{\epsilon^\mu}{8}\left(\epsilon_{\mu\sigma\tau\nu} + g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma}\right) D^\nu\bar\phi + v^\nu\partial_\nu\chi_{\sigma\tau}\,,\\
\mathcal{Q}b_{\mu\nu} &= [c,b_{\mu\nu}] + \omega(-(D_\mu\psi_\nu - D_\nu\psi_\mu)^{-} + [\chi_{\mu\nu},\phi])\\
&\quad -\Big(\epsilon_\mu(D^\tau\chi_{\nu\tau} - D^\tau\chi_{\tau\nu} + \epsilon_{\nu\gamma\lambda\tau}D^\gamma\chi^{\lambda\tau}) + \epsilon_\mu D_\nu\eta - \epsilon_\mu[\psi_\nu,\bar\phi]\Big)^{-} + v^\rho\partial_\rho b_{\mu\nu}\,,\\
\mathcal{Q}\eta &= \{c,\eta\} + \frac{\omega}{2}[\phi,\bar\phi] + \frac{\epsilon^\mu}{2}D_\mu\bar\phi + v^\nu\partial_\nu\eta\,,\\
\mathcal{Q}\phi &= [c,\phi] - \epsilon^\mu\psi_\mu + v^\nu\partial_\nu\phi\,,\\
\mathcal{Q}\bar\phi &= [c,\bar\phi] + 2\omega\eta + v^\nu\partial_\nu\bar\phi\,,\\
\mathcal{Q}c &= c^2 - \omega^2\phi - \epsilon^\mu A_\mu + \frac{\epsilon^2}{16}\bar\phi + v^\nu\partial_\nu c\,,\\
\mathcal{Q}v^\mu &= -\omega\epsilon^\mu\,,\\
\mathcal{Q}\epsilon^\mu &= 0\,,\\
\mathcal{Q}\omega &= 0\,,\\
\mathcal{Q}\bar c &= b + v^\mu\partial_\mu\bar c\,,\\
\mathcal{Q}b &= \omega\epsilon^\mu\partial_\mu\bar c + v^\mu\partial_\mu b\,.
\end{aligned}
\tag{A.1}
$$

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